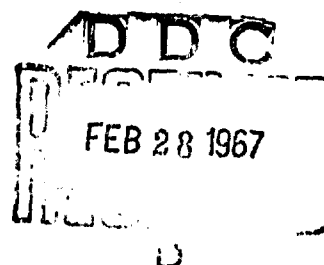
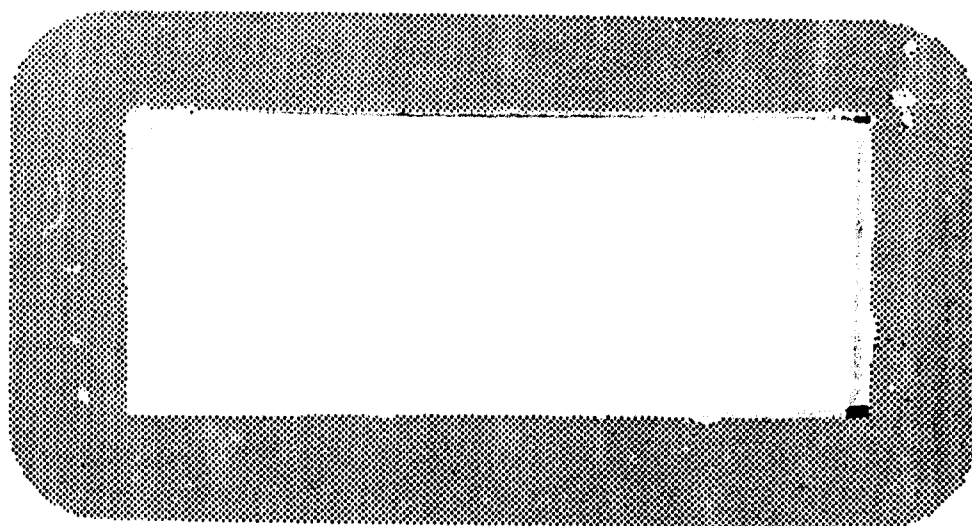


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TECHNICAL MEMORANDUM



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U. S. NAVAL WEAPONS LABORATORY

TECHNICAL MEMORANDUM

December 1961

No. K-34/61

THE USE OF DIFFERENCE METHODS IN
MAKING POLYNOMIAL FITS TO DISCRETE DATA

by

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This memorandum is not to be construed as expressing the opinion of the U. S. Naval Weapons Laboratory, and while its contents are considered correct, they are subject to modification upon further study.

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ABSTRACT

A difference method, which for one choice of weights applied to the differences leads to equally weighted least squares polynomial fits to equally spaced data, is generalized so that the approximation can be obtained in terms of polynomials chosen by the user in the fitting of equally spaced or unequally spaced data.

FOREWORD

Using this method, an investigation was made into the ballistics field with particular emphasis on the preparation of bombing tables. The work was conducted under Task Assignment Number RMMO-42007/2101/FO0809001. Credit is due Richard Hageman, Richard Shannon, Gordon Barker, and Mrs. Jane H. Martina for their assistance in the preparation of the tables found in Appendix A and the example in Appendix B.

INTRODUCTION

The technique of using differences in making polynomial fits to discrete data is not new and has been widely discussed in the literature. However, the method described in this report enables the user to apply this method using any form of interpolating polynomial which best suits his purpose. Also, by changing from ordinary differences, which are used in working with equally spaced data, to divided differences, the method can be used for unequally spaced data.

For the convenience of the user Appendix A contains the tabulations of the multipliers, the weights and the summations of weights through the 4th difference for equally spaced data containing from 5 to 30 points. With the modification presented in the text on Page 9 these weights may be adapted for use with unequally spaced data.

EQUALLY SPACED DATA

Exact nth Degree Fit to $n + 1$ Points

Newton, Gauss, and others have given formulas for obtaining an nth degree fit to $n + 1$ equally spaced points (see reference (a)). Such formulas have the advantage of being readily available, but the disadvantage of not giving the result in the form desired for many applications. At the expense of additional computation it is possible to get the interpolating polynomial in the desired form for any application.

Suppose this form for the n th degree polynomial through $n + 1$ points is

$$f_n = A_0 P_0 + A_1 P_1 + A_2 P_2 + \dots + A_n P_n \quad (1)$$

P_k can be any k th degree polynomial. The argument used in P_k can be the original independent variable, x , or some other argument, say u , which is a linear function of x .

If the $n + 1$ points are substituted in equation (1), $n + 1$ equations in A_0, A_1, \dots, A_n are obtained. Since

$$\begin{aligned} \Delta^n P_j &= 0 \quad \text{for } j < n, \\ \Delta^n f_n &= A_n \Delta^n P_n. \end{aligned} \quad (2)$$

However,

$$\Delta^n P_n = h^n P_n^{(n)} \quad (3)$$

where $P_n^{(r)}$ is the r th derivative of P_n with respect to the argument used, and h is the increment in that argument between successive points. Equation (2) can then be solved for A_n , with

$$A_n = \frac{\Delta^n f_n}{\Delta^n P_n} = \frac{\Delta^n f_n}{h^n P_n^{(n)}}.$$

Then

$$f_{n-1} = f_n - \frac{\Delta^n f_n}{h^n P_n^{(n)}} P_n \quad (4)$$

will be a polynomial of degree $n - 1$.

If the first n values of f_{n-1} are used it is now possible to find

$$A_{n-1} = \frac{\Delta^{n-1} f_{n-1}}{h^{n-1} P_{n-1}^{(n-1)}}, \quad (5)$$

and

$$f_{n-2} = f_{n-1} - \frac{\Delta^{n-1} f_{n-1}}{h^{n-1} P_{n-1}^{(n-1)}} P_{n-1} \quad (6)$$

will be a polynomial of degree $n - 2$. The first $n - 1$ values of f_{n-2} can be used to find A_{n-2} and so on until all the A 's have been evaluated. The Newton formula uses P_k 's chosen in such a way that P_n vanishes for the n points used in finding A_{n-1} , f_n and P_{n-1} vanish for the $n - 1$ points used in finding A_{n-2} , etc., which is ideal provided the form obtained for f_n is convenient for use. For some purposes it is desirable to have $P_k = x^k$. It is obvious in such a case that the derivatives and integrals may be seen at a glance if necessary.

Another purpose might call for the use of the Tchebychev polynomials on the right hand side of equation (1) if truncation to a lower degree appears desirable. Other uses of f_n will demand other forms of P_k .

Smoothed nth Degree Fit to $n + 1 + m$ Points

Let it be assumed that the function defined by the $n + 1 + m$ values is

$$f = f_n + e \quad (7)$$

where f_n is an n th degree polynomial and e is the error of observation or noise. If a satisfactory method for computing $(\Delta^k y)$, the mean k th difference, is available and it can be assumed that $(\Delta^k e)$ will be sufficiently small for all k , then the approximation of f ,

$$f_n = A_0 P_0 + A_1 P_1 + A_2 P_2 + \dots + A_n P_n,$$

can be found.

Since e is involved it will be necessary to use $n + 1 + m$ values in the computation of every mean difference. A_n can be found from

$$(\Delta^n f) \approx A_n \Delta^n P_n = A_n \left(h^n P_n^{(n)} \right). \quad (8)$$

Let

$$\begin{aligned} \hat{f}_n &= f = f_n + e \\ \hat{f}_{n-1} &= \hat{f}_n - A_n P_n = f_{n-1} + e \\ \hat{f}_{n-2} &= \hat{f}_{n-1} - A_{n-1} P_{n-1} = f_{n-2} + e, \quad \text{etc.} \end{aligned}$$

Then, for each A_k ,

$$A_k = \frac{\overline{\Delta^k f_k}}{\Delta^k P_k} - \frac{\overline{\Delta^k \hat{f}_k}}{h^k P_k^{(k)}} \quad (9)$$

Finding Mean Differences

The weighted mean of the first difference of a set of equally spaced y's can be written

$$(\bar{\Delta y}) = \frac{1}{\sum w} [w_1 w_2 \dots w_r] \begin{bmatrix} -1 & 1 & 0 & 0 & \dots & . & 0 \\ 0 & -1 & 1 & 0 & \dots & . & 0 \\ 0 & 0 & -1 & 1 & \dots & . & 0 \\ . & . & . & . & \dots & . & . \\ . & . & . & . & \dots & . & . \\ . & . & . & . & \dots & . & . \\ 0 & 0 & 0 & 0 & \dots & -1 & 1 \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ . \\ . \\ . \\ y_r \end{bmatrix} \quad (10)$$

If equation (10) is written in the expanded form

$$(\bar{\Delta y}) = \frac{1}{\sum w} (m_0 y_0 + m_1 y_1 + m_2 y_2 + \dots + m_r y_r) \quad (11)$$

and in the form

$$(\bar{\Delta y}) = \frac{1}{\sum w} [0 w_1 w_2 \dots w_r 0] \begin{bmatrix} 1 & 0 & 0 & \dots & . & 0 \\ -1 & 1 & 0 & \dots & . & 0 \\ 0 & -1 & 1 & \dots & . & 0 \\ . & . & . & \dots & . & . \\ . & . & . & \dots & . & . \\ . & . & . & \dots & . & . \\ 0 & 0 & 0 & \dots & -1 & 1 \\ 0 & 0 & 0 & \dots & 0 & -1 \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ . \\ . \\ . \\ y_r \end{bmatrix} \quad (12)$$

comparison of equations (11) and (12) shows that the m's are minus the first differences of the set $[0w_1w_2w_3\ldots w_r0]$. Given the w's, the m's can be found and given the m's, the w's can be found. All the weighted mean first differences can be found by these formulas.

There are many ways in which the m's (or w's) can be chosen. One method of making the choice is to minimize the maximum absolute value

of $\frac{m_1}{\sum w}$ (see reference (b)). A choice which is more consistent with

practice is to assume the individual errors to be uncorrelated and to have equal variance. With this in mind let the w's be chosen in such a way as to minimize $M = \sum m^2$, holding $W = \sum w$ constant, which will minimize the variance of (\bar{Y}) .

Then

$$dM = 2 \left\{ \left[m_0 \frac{\partial m_0}{\partial w_1} + m_1 \frac{\partial m_1}{\partial w_1} + \dots \right] dw_1 \right. \\ + \left[m_0 \frac{\partial m_0}{\partial w_2} + m_1 \frac{\partial m_1}{\partial w_2} + m_2 \frac{\partial m_2}{\partial w_2} + \dots \right] dw_2 \\ + \dots \\ \left. + \left[\dots + m_{r-2} \frac{\partial m_{r-2}}{\partial w_r} + m_{r-1} \frac{\partial m_{r-1}}{\partial w_r} + m_r \frac{\partial m_r}{\partial w_r} \right] dw_r \right\} = 0 \quad , \quad (13)$$

$$dW = dw_1 + dw_2 + \dots + dw_r = 0 \quad . \quad (14)$$

The requirement for a constrained minimum is met if the rank of the matrix of the coefficients of the dw's in equations (13) and (14) is one. Therefore, the coefficients of the dw's in equation (13) are all equal. From equations (11) and (12) each of these coefficients is a constant times a first difference of the m's. The m's, in turn, are minus the first differences of the set

$$[0w_1w_2\ldots w_r0]$$

so that this set has a constant second difference and the w function is a quadratic with two known zeros.

The weighted mean of the second difference can be written:

$$(\overline{\Delta^2 y}) = \frac{1}{\sum w} [00w_1w_2\dots w_{r-1}00]$$

1	0	0	0	0	0
-2	1	0	0	0	0
1	-2	1	0	0	0
0	1	-2	1	0	0
.
.
.
0	0	0	0	0	..1	-2	1
0	0	0	0	0	..0	1	-2
0	0	0	0	0	..0	0	1

y_0
y_1
.
.
.
y_r

or

$$(\overline{\Delta^2 y}) = \frac{1}{\sum w} (m_0 y_0 + m_1 y_1 + m_2 y_2 + \dots + m_N y_r) \quad (15)$$

The m 's are the second differences of the set

$$[00w_1w_2\dots w_{r-1}00]$$

and minimizing $\sum m^2$ with $\sum w$ held constant leads to the requirement that the w function be a fourth degree polynomial with four known zeros.

In finding the mean k th difference, $(\overline{\Delta^k y})$, the m 's will be $(-1)^k$ times the k th difference of the set

$$[00\dots 0w_1w_2\dots w_{r-k+1}0\dots 00]$$

having k zeros at each end. The requirement that $\sum m^2$ be a minimum with $\sum w$ held constant leads to a $2k$ th degree polynomial with $2k$ known zeros for the w function.

The i th weight in getting the mean k th difference from $r + 1$ points can thus be written

$$\rho w_i = \binom{k - 1 + i}{k} \binom{r + 1 - i}{k}, \quad (16)$$

where ρ is some arbitrary constant. The weights given in Table 2 in Appendix A were obtained by using equation (16). It is seen from the equation that for $k = 0$, $\rho w_i \equiv 1$ for all i .

Since the m 's for finding the mean k th difference satisfy a k th degree polynomial and since the $(k + j)$ th difference of a k th degree polynomial is zero for $j > 0$, the m polynomials are orthogonal over an equally spaced set with equal weighting and are identical, except for an arbitrary multiplier, with orthogonal polynomials for equal spacing and equal weighting discussed on pages 287 - 291 of reference (a) and given on pages 375 - 381 of reference (c). The use of equation (16) in finding the mean k th difference thus leads to an equally weighted least squares polynomial fit of equally spaced data for all possible choices of the P_k 's in equation (1).

The results obtained using the w polynomials of equation (16) and the resulting m polynomials for the P_k 's in equation (1) are, of course, very familiar. The use of the w 's to fit in terms of general P_k 's and the modification of the w 's for fitting unequally spaced data are not so familiar. More general methods for choosing the w 's and m 's may well be a rewarding field for further study.

UNEQUALLY SPACED DATA

Exact n th Degree Fit to $n + 1$ Points

The method for equally spaced data may be modified for use with unequally spaced data by using divided differences instead of ordinary differences.

Thus, if

$$f_n = A_0 P_0 + A_1 P_1 + A_2 P_2 + \dots + A_n P_n, \quad (1)$$

$$f[u_0, u_1, \dots, u_n] = A_n P_n[u_0, u_1, \dots, u_n] \quad (17)$$

where

$$\begin{aligned}
 f[u_0, u_1, \dots, u_n] &= \frac{f(u_0)}{(u_0 - u_1) \dots (u_0 - u_n)} \\
 &+ \frac{f(u_1)}{(u_1 - u_0) \dots (u_1 - u_n)} \\
 &+ \dots \\
 &+ \frac{f(u_n)}{(u_n - u_0) \dots (u_n - u_{n-1})} ,
 \end{aligned}$$

for any positive integer n , is the n th divided difference defined in reference (a).

If

$$P_n(u) = B_0 + B_1u + B_2u^2 + \dots + B_nu^n ,$$

the n th divided difference of $P_n(u)$ is

$$P_n[u_0, u_1, u_2, \dots, u_n] = B_n .$$

So that, except for the arithmetic required in the computation of $f[u_0, u_1, u_2, \dots, u_n]$, A_n is easily found from equation (17).

The finding of f_{n-1} , f_{n-2} , ..., and A_{n-1} , A_{n-2} , ... follows a path parallel to that described for equally spaced data.

Smoothed n th Degree Fit to $n + 1 + m$ Points

Let

$$(\overline{\Delta^k} f)$$

be the mean k th divided difference of f . Then equation (9) can be rewritten

$$A_k = \frac{(\overline{\Delta^k} f_k)}{\overline{\Delta^k} P_k} , \quad (18)$$

and the problem of fitting is then solved if a satisfactory way can be found for finding the mean k th divided difference.

Finding Mean Divided Differences

The mean kth divided difference can be found from

$$(\Delta^k f) = \frac{1}{\sum \omega} \left\{ \omega_1 f[u_0, u_1, \dots, u_k] + \omega_2 f[u_1, u_2, \dots, u_{k+1}] + \dots \right\} \quad (19)$$

The optimum choice of the ω 's is a more difficult problem than was the choice of the w 's in the case of equally spaced data. A possible choice is to take

$$\omega_1 = (u_k - u_0) w_1$$

$$\omega_2 = (u_{k+1} - u_1) w_2$$

$$\begin{array}{c} \cdot \\ \cdot \\ \cdot \end{array} \quad \begin{array}{c} \cdot \\ \cdot \\ \cdot \end{array}$$

$$\omega_i = (u_{k+i+1} - u_{i+1}) w_i$$

$$\begin{array}{c} \cdot \\ \cdot \\ \cdot \end{array} \quad \begin{array}{c} \cdot \\ \cdot \\ \cdot \end{array}$$

the w 's being those chosen for finding the mean kth difference for equally spaced data. For finding the differences when $k = 0$, choose $\omega_1 = 1$.

This will approach the equally spaced fitting as the intervals become more nearly equal and may be expected to give reasonable results in other cases.

It should be mentioned that the interesting difference relation between the w 's and m 's, and the resulting generation of the m polynomials, for equally spaced data does not hold true for the ω 's and m 's used with unequally spaced data involving divided differences.

Given in Appendix B is an example using this divided difference technique in smoothing $y = \sin x$ over a non-constant data interval.

REFERENCES

- (a) Hildebrand, F. B., Introduction to Numerical Analysis, McGraw-Hill, 1956
- (b) NWL Technical Memorandum No. K-12/61, Smoothing and Differentiation of Tracking Data Taken Across Separation Impulse, May 1961
- (c) Milne, W. E., Numerical Calculus, Princeton University Press, 1949

APPENDIX A

TABLE 1
MULTIPLIERS FOR EQUALLY SPACED DATA

5 POINTS

i	m(0)	m(1)	m(2)	m(3)	m(4)
0	1	-2	2	-1	1
1	1	-1	-1	2	-4
2	1	0	-2	0	6
3	1	1	-1	-2	-4
4	1	2	2	1	1
$\sum w$	5	10	7	2	1
$\sum (m)^2$	5	10	14	10	70

6 POINTS

i	m(0)	m(1)	m(2)	m(3)	m(4)
0	1	-5	5	-5	1
1	1	-3	-1	7	-3
2	1	-1	-4	4	2
3	1	1	-4	-4	2
4	1	3	-1	-7	-3
5	1	5	5	5	1
$\sum w$	6	35	28	18	2
$\sum (m)^2$	6	70	84	180	28

TABLE 1 (Continued)

7 POINTS

i	m(0)	m(1)	m(2)	m(3)	m(4)
0	1	-3	5	-1	3
1	1	-2	0	1	-7
2	1	-1	-3	1	1
3	1	0	-4	0	6
4	1	1	-3	-1	1
5	1	2	0	-1	-7
6	1	3	5	1	3
Σw	7	28	42	6	11
$\Sigma(m)^2$	7	28	84	6	154

8 POINTS

i	m(0)	m(1)	m(2)	m(3)	m(4)
0	1	-7	7	-7	7
1	1	-5	1	5	-13
2	1	-3	-3	7	-3
3	1	-1	-5	3	9
4	1	1	-5	-3	9
5	1	3	-3	-7	-3
6	1	5	1	-5	-13
7	1	7	7	7	7
Σw	8	84	84	66	44
$\Sigma(m)^2$	8	168	168	264	616

TABLE 1 (Continued)

9 POINTS

1	m(0)	m(1)	m(2)	m(3)	m(4)
0	1	-4	28	-14	14
1	1	-3	7	7	-21
2	1	-2	-8	13	-11
3	1	-1	-17	9	9
4	1	0	-20	0	18
5	1	1	-17	-9	9
6	1	2	-8	-13	-11
7	1	3	7	-7	-21
8	1	4	28	14	14
Σw	9	60	462	198	143
$\Sigma (m)^2$	9	60	2772	990	2002

10 POINTS

1	m(0)	m(1)	m(2)	m(3)	m(4)
0	1	-9	6	-42	18
1	1	-7	2	14	-22
2	1	-5	-1	35	-17
3	1	-3	-3	31	3
4	1	-1	-4	12	18
5	1	1	-4	-12	18
6	1	3	-3	-31	3
7	1	5	-1	-35	-17
8	1	7	2	-14	-22
9	1	9	6	42	18
Σw	10	165	132	858	286
$\Sigma (m)^2$	10	330	132	8580	2860

TABLE 1 (Continued)

11 POINTS

1	m(0)	m(1)	m(2)	m(3)	m(4)
0	1	-5	15	-30	6
1	1	-4	6	6	-6
2	1	-3	-1	22	-6
3	1	-2	-6	23	-1
4	1	-1	-9	14	4
5	1	0	-10	0	6
6	1	1	-9	-14	4
7	1	2	-6	-23	-1
8	1	3	-1	-22	-6
9	1	4	6	-6	-6
10	1	5	15	30	6
$\sum w$	11	110	429	858	143
$\sum (m)^2$	11	110	858	4290	286

12 POINTS

1	m(0)	m(1)	m(2)	m(3)	m(4)
0	1	-11	55	-33	33
1	1	-9	25	3	-27
2	1	-7	1	21	-33
3	1	-5	-17	25	-13
4	1	-3	-29	19	12
5	1	-1	-35	7	28
6	1	1	-35	-7	28
7	1	3	-29	-19	12
8	1	5	-17	-25	-13
9	1	7	1	-21	-33
10	1	9	25	-3	-27
11	1	11	55	33	33
$\sum w$	12	286	2002	1287	1144
$\sum (m)^2$	12	572	12,012	5148	8008

TABLE 1 (Continued)

13 POINTS

i	m(0)	m(1)	m(2)	m(3)	m(4)
0	1	-6	22	-11	99
1	1	-5	11	0	-66
2	1	-4	2	6	-96
3	1	-3	-5	8	-54
4	1	-2	-10	7	11
5	1	-1	-13	4	64
6	1	0	-14	0	84
7	1	1	-13	-4	64
8	1	2	-10	-7	11
9	1	3	-5	-8	-54
10	1	4	2	-6	-96
11	1	5	11	0	-66
12	1	6	22	11	99
$\sum w$	13	182	1001	572	4862
$\sum (m)^2$	13	182	2002	572	68,068

TABLE 1 (Continued)

14 POINTS

1	m(0)	m(1)	m(2)	m(3)	m(4)
0	1	-13	13	-143	143
1	1	-11	7	-11	-77
2	1	-9	2	66	-132
3	1	-7	-2	96	-92
4	1	-5	-5	95	-13
5	1	-3	-7	67	63
6	1	-1	-8	24	108
7	1	1	-8	-24	108
8	1	3	-7	-67	63
9	1	5	-5	-95	-13
10	1	7	-2	-96	-92
11	1	9	2	-66	-132
12	1	11	7	11	-77
13	1	13	13	143	143
$\sum w$	14	455	728	9724	9724
$\sum (-)$	14	910	728	9724	136,136

TABLE 1 (Continued)

15 POINTS

1	m(0)	m(1)	m(2)	m(3)	m(4)
0	1	-7	91	-91	1001
1	1	-6	52	-13	-429
2	1	-5	19	35	-869
3	1	-4	-8	58	-704
4	1	-3	-29	61	-249
5	1	-2	-44	49	251
6	1	-1	-53	27	621
7	1	0	-56	0	756
8	1	1	-53	-27	621
9	1	2	-44	-49	251
10	1	3	-29	-61	-249
11	1	4	-8	-58	-704
12	1	5	19	-35	-869
13	1	6	52	13	-429
14	1	7	91	91	1001
$\sum w$	15	280	6188	7956	92,378
$\sum (m)^2$	15	280	37,128	39,780	6,466,460

TABLE 1 (Continued)

16 POINTS

1	m(0)	m(1)	m(2)	m(3)	m(4)
0	1	-15	35	-455	1365
1	1	-13	21	-91	-455
2	1	-11	9	143	-1105
3	1	-9	-1	267	-1005
4	1	-7	-9	301	-505
5	1	-5	-15	265	115
6	1	-3	-19	179	645
7	1	-1	-21	63	945
8	1	1	-21	-63	945
9	1	3	-19	-179	645
10	1	5	-15	-265	115
11	1	7	-9	-301	-505
12	1	9	-1	-267	-1005
13	1	11	9	-143	-1105
14	1	13	21	91	-455
15	1	15	35	455	1365
Σw	16	680	2856	50,388	167,960
$\Sigma (m)^2$	16	1360	5712	1,007,760	11,757,200

TABLE 1 (Continued)

17 POINTS

1	m(0)	m(1)	m(2)	m(3)	m(4)
0	1	-8	40	-28	52
1	1	-7	25	-7	-13
2	1	-6	12	7	-39
3	1	-5	1	15	-39
4	1	-4	-8	18	-24
5	1	-3	-15	17	-3
6	1	-2	-20	13	17
7	1	-1	-23	7	31
8	1	0	-24	0	36
9	1	1	-23	-7	31
10	1	2	-20	-13	17
11	1	3	-15	-17	-3
12	1	4	-8	-18	-24
13	1	5	1	-15	-39
14	1	6	12	-7	-39
15	1	7	25	7	-13
16	1	8	40	28	52
Σw	17	408	3876	3876	8398
$\Sigma(m)^2$	17	408	7752	3876	16,796

TABLE 1 (Continued)

18 POINTS

1	m(0)	m(1)	m(2)	m(3)	m(4)
0	1	-17	68	-68	68
1	1	-15	44	-20	-12
2	1	-13	23	13	-47
3	1	-11	5	33	-51
4	1	-9	-10	42	-36
5	1	-7	-22	42	-12
6	1	-5	-31	35	13
7	1	-3	-37	23	33
8	1	-1	-40	8	44
9	1	1	-40	-8	44
10	1	3	-37	-23	33
11	1	5	-31	-35	13
12	1	7	-22	-42	-12
13	1	9	-10	-42	-36
14	1	11	5	-33	-51
15	1	13	23	-13	-47
16	1	15	44	20	-12
17	1	17	68	68	68
Σw	18	969	7752	11,628	14,212
$\Sigma (m)^2$	18	1938	23,256	23,256	28,424

TABLE 1 (Continued)

19 POINTS

1	m(0)	m(1)	m(2)	m(3)	m(4)
0	1	-9	51	-204	612
1	1	-8	34	-68	-68
2	1	-7	19	28	-388
3	1	-6	6	89	-453
4	1	-5	-5	120	-354
5	1	-4	-14	126	-168
6	1	-3	-21	112	42
7	1	-2	-26	83	227
8	1	-1	-29	44	352
9	1	0	-30	0	396
10	1	1	-29	-44	352
11	1	2	-26	-83	227
12	1	3	-21	-112	42
13	1	4	-14	-126	-168
14	1	5	-5	-120	-354
15	1	6	6	-89	-453
16	1	7	19	-28	-388
17	1	8	34	68	-68
18	1	9	51	204	612
Σw	19	570	6783	42,636	163,438
$\Sigma (m)^2$	19	570	13,566	213,180	2,288,132

TABLE 1 (Continued)

20 POINTS

i	m(0)	m(1)	m(2)	m(3)	m(4)
0	1	-19	57	-969	1938
1	1	-17	39	-357	-102
2	1	-15	23	85	-1122
3	1	-13	9	377	-1402
4	1	-11	-3	539	-1187
5	1	-9	-13	591	-687
6	1	-7	-21	553	-77
7	1	-5	-27	445	503
8	1	-3	-31	287	948
9	1	-1	-33	99	1188
10	1	1	-33	-99	1188
11	1	3	-31	-287	948
12	1	5	-27	-445	503
13	1	7	-21	-553	-77
14	1	9	-13	-591	-687
15	1	11	-3	-539	-1187
16	1	13	9	-377	-1402
17	1	15	23	-85	-1122
18	1	17	39	357	-102
19	1	19	57	969	1238
Σw	20	1330	8778	245,157	653,752
$\Sigma G_n)^2$	20	2660	17,556	4,903,140	22,881,320

TABLE 1 (Continued)

21 POINTS

i	m(0)	m(1)	m(2)	m(3)	m(4)
0	1	-10	190	-285	969
1	1	-9	133	-114	0
2	1	-8	82	12	-510
3	1	-7	37	98	-680
4	1	-6	-2	149	-615
5	1	-5	-35	170	-406
6	1	-4	-62	166	-130
7	1	-3	-83	142	150
8	1	-2	-98	103	385
9	1	-1	-107	54	540
10	1	0	-110	0	594
11	1	1	-107	-54	540
12	1	2	-98	-103	385
13	1	3	-83	-142	150
14	1	4	-62	-166	-130
15	1	5	-35	-170	-406
16	1	6	-2	-149	-615
17	1	7	37	-98	-680
18	1	8	82	-12	-510
19	1	9	133	114	0
20	1	10	190	285	969
Σw	21	770	33,649	86,526	408,595
$\Sigma (m)^2$	21	770	201,894	432,630	5,720,330

TABLE 1 (Continued)

22 POINTS

i	m(0)	m(1)	m(2)	m(3)	m(4)
0	1	-21	35	-133	1197
1	1	-19	25	-57	57
2	1	-17	16	0	-570
3	1	-15	8	40	-810
4	1	-13	1	65	-775
5	1	-11	-5	77	-563
6	1	-9	-10	78	-258
7	1	-7	-14	70	70
8	1	-5	-17	55	365
9	1	-3	-19	35	585
10	1	-1	-20	12	702
11	1	1	-20	-12	702
12	1	3	-19	-35	585
13	1	5	-17	-55	365
14	1	7	-14	-70	70
15	1	9	-10	-78	-258
16	1	11	-5	-77	-563
17	1	13	1	-65	-775
18	1	15	8	-40	-810
19	1	17	16	0	-570
20	1	19	25	57	57
21	1	21	35	133	1197
Σw	22	1771	7084	48,070	624,910
$\Sigma (m)^2$	22	3542	7084	96,140	8,748,740

TABLE 1 (Continued)

23 POINTS

1	m(0)	m(1)	m(2)	m(3)	m(4)
0	1	-11	77	-77	1463
1	1	-10	56	-35	133
2	1	-9	37	-3	-627
3	1	-8	20	20	-950
4	1	-7	5	35	-955
5	1	-6	-8	43	-747
6	1	-5	-19	45	-417
7	1	-4	-28	42	-42
8	1	-3	-35	35	315
9	1	-2	-40	25	605
10	1	-1	-43	13	793
11	1	0	-44	0	858
12	1	1	-43	-13	793
13	1	2	-40	-25	605
14	1	3	-35	-35	315
15	1	4	-28	-42	-42
16	1	5	-19	-45	-417
17	1	6	-8	-43	-747
18	1	7	5	-35	-955
19	1	8	20	-20	-950
20	1	9	37	3	-627
21	1	10	56	35	133
22	1	11	77	77	1463
Σw	23	1012	17,710	32,890	937,365
$\Sigma (m)^2$	23	1012	35,420	32,890	13,123,110

TABLE 1 (Continued)

24 POINTS

1	m(0)	m(1)	m(2)	m(3)	m(4)
0	1	-23	253	-1771	253
1	1	-21	187	-847	33
2	1	-19	127	-133	-97
3	1	-17	73	391	-157
4	1	-15	25	745	-165
5	1	-13	-17	949	-137
6	1	-11	-53	1023	-87
7	1	-9	-83	987	-27
8	1	-7	-107	861	33
9	1	-5	-125	665	85
10	1	-3	-137	419	123
11	1	-1	-143	143	143
12	1	1	-143	-143	143
13	1	3	-137	-419	123
14	1	5	-125	-665	85
15	1	7	-107	-861	33
16	1	9	-83	-987	-27
17	1	11	-53	-1023	-87
18	1	13	-17	-949	-137
19	1	15	25	-745	-165
20	1	17	73	-391	-157
21	1	19	127	133	-97
22	1	21	187	847	33
23	1	23	253	1771	253
Σw	24	2300	65,780	888,030	197,340
$\Sigma (m)^2$	24	4600	394,680	17,760,600	394,680

TABLE 1 (Continued)

25 POINTS

i	m(0)	m(1)	m(2)	m(3)	m(4)
0	1	-12	02	-506	1518
1	1	-11	69	-253	253
2	1	-10	40	-55	-517
3	1	-9	29	93	-897
4	1	-8	12	196	-982
5	1	-7	-3	259	-857
6	1	-6	-16	287	-597
7	1	-5	-27	285	-267
8	1	-4	-36	258	78
9	1	-3	-43	211	393
10	1	-2	-48	149	643
11	1	-1	-51	77	803
12	1	0	-52	0	858
13	1	1	-51	-77	803
14	1	2	-48	-149	643
15	1	3	-43	-211	393
16	1	4	-36	-258	78
17	1	5	-27	-285	-267
18	1	6	-16	-287	-597
19	1	7	-3	-259	-857
20	1	8	12	-196	-982
21	1	9	29	-93	-897
22	1	10	48	55	-517
23	1	11	69	253	253
24	1	12	02	506	1518
Σw	25	1300	26,010	296,010	1,430,715
$\Sigma (m)^2$	25	1300	53,820	1,480,050	14,307,150

TABLE 1 (Continued)

26 POINTS

i	m(0)	m(1)	m(2)	m(3)	m(4)
0	1	-25	50	-3450	2530
1	1	-23	38	-1794	506
2	1	-21	27	-483	-759
3	1	-19	17	513	-1419
4	1	-17	8	1224	-1614
5	1	-15	0	1680	-1470
6	1	-13	-7	1911	-1099
7	1	-11	-13	1947	-599
8	1	-9	-18	1818	-54
9	1	-7	-22	1554	466
10	1	-5	-25	1185	905
11	1	-3	-27	741	1221
12	1	-1	-28	252	1386
13	1	1	-28	-252	1386
14	1	3	-27	-741	1221
15	1	5	-25	-1185	905
16	1	7	-22	-1554	466
17	1	9	-18	-1818	-54
18	1	11	-13	-1947	-599
19	1	13	-7	-1911	-1099
20	1	15	0	-1680	-1470
21	1	17	8	-1224	-1614
22	1	19	17	-513	-1419
23	1	21	27	483	-759
24	1	23	38	1794	506
25	1	25	50	3450	2530
Σw	26	2925	16,380	2,341,170	2,861,430
$\Sigma(m)^2$	26	5850	16,380	70,235,100	40,060,020

TABLE 1 (Continued)

27 POINTS

1	m(0)	m(1)	m(2)	m(3)	m(4)
0	1	-13	325	-130	2990
1	1	-12	250	-70	690
2	1	-11	181	-22	-782
3	1	-10	118	15	-1587
4	1	-9	61	42	-1872
5	1	-8	10	60	-1770
6	1	-7	-35	70	-1400
7	1	-6	-74	73	-867
8	1	-5	-107	70	-262
9	1	-4	-134	62	333
10	1	-3	-155	50	870
11	1	-2	-170	35	1285
12	1	-1	-179	18	1548
13	1	0	-182	0	1638
14	1	1	-179	-18	1548
15	1	2	-170	-35	1285
16	1	3	-155	-50	870
17	1	4	-134	-62	333
18	1	5	-107	-70	-262
19	1	6	-74	-73	-867
20	1	7	-35	-70	-1400
21	1	8	10	-60	-1770
22	1	9	61	-42	-1872
23	1	10	118	-15	-1587
24	1	11	181	22	-782
25	1	12	250	70	690
26	1	13	325	130	2990
Σw	27	1638	118,755	101,790	4,032,015
$\Sigma (m)^2$	27	1638	712,530	101,790	56,448,210

TABLE 1 (Continued)

28 POINTS

i	m (0)	m (1)	m (2)	m (3)	m (4)
0	1	-27	117	-1755	1755
1	1	-25	91	-975	455
2	1	-23	67	-345	-395
3	1	-21	45	147	-879
4	1	-19	25	513	-1074
5	1	-17	7	765	-1050
6	1	-15	-9	915	-870
7	1	-13	-23	975	-590
8	1	-11	-35	957	-259
9	1	-9	-45	873	81
10	1	-7	-53	735	395
11	1	-5	-59	555	655
12	1	-3	-63	345	840
13	1	-1	-65	117	936
14	1	1	-65	-117	936
15	1	3	-63	-345	840
16	1	5	-59	-555	655
17	1	7	-53	-735	395
18	1	9	-45	-873	81
19	1	11	-35	-957	-259
20	1	13	-23	-975	-590
21	1	15	-9	-915	-870
22	1	17	7	-765	-1050
23	1	19	25	-513	-1074
24	1	21	45	-147	-879
25	1	23	67	345	-395
26	1	25	91	975	455
27	1	27	117	1755	1755
$\sum w$	28	3654	47,502	1,577,745	2,804,880
$\sum (m)^2$	28	7308	95,004	19,342,940	19,034,160

TABLE 1 (Continued)

29 POINTS

i	m(0)	m(1)	m(2)	m(3)	m(4)
0	1	-14	126	-819	4095
1	1	-13	99	-468	1170
2	1	-12	74	-182	-780
3	1	-11	51	44	-1930
4	1	-10	30	215	-2441
5	1	-9	11	336	-2460
6	1	-8	-6	412	-2120
7	1	-7	-21	448	-1540
8	1	-6	-34	449	-825
9	1	-5	-45	420	-66
10	1	-4	-54	366	660
11	1	-3	-61	292	1290
12	1	-2	-66	203	1775
13	1	-1	-69	104	2080
14	1	0	-70	0	2184
15	1	1	-69	-104	2080
16	1	2	-66	-203	1775
17	1	3	-61	-292	1290
18	1	4	-54	-366	660
19	1	5	-45	-420	-66
20	1	6	-34	-449	-825
21	1	7	-21	-448	-1540
22	1	8	-6	-412	-2120
23	1	9	11	-336	-2460
24	1	10	30	-215	-2441
25	1	11	51	-44	-1930
26	1	12	74	182	-780
27	1	13	99	468	1170
28	1	14	126	819	4095
$\sum w$	29	2030	56,637	841,464	7,713,420
$\sum (m)^2$	29	2030	113,274	4,207,320	107,987,880

TABLE 1 (Continued)

30 POINTS

i	m(0)	m(1)	m(2)	m(3)	m(4)
0	1	-29	203	-1827	23751
1	1	-27	161	-1071	7371
2	1	-25	122	-450	-3744
3	1	-23	86	46	-10504
4	1	-21	53	427	-13749
5	1	-19	23	703	-14249
6	1	-17	-4	884	-12704
7	1	-15	-28	980	-9744
8	1	-13	-49	1001	-5929
9	1	-11	-67	957	-1749
10	1	-9	-82	858	2376
11	1	-7	-94	714	6069
12	1	-5	-103	535	9131
13	1	-3	-109	331	11271
14	1	-1	-112	112	12376
15	1	1	-112	-112	12376
16	1	3	-109	-331	11271
17	1	5	-103	-535	9131
18	1	7	-94	-714	6069
19	1	9	-82	-858	2376
20	1	11	-67	-957	-1749
21	1	13	-49	-1001	-5929
22	1	15	-28	-980	-9744
23	1	17	-4	-884	-12704
24	1	19	23	-703	-14249
25	1	21	53	-427	-13749
26	1	23	86	-46	-10504
27	1	25	122	450	-3744
28	1	27	161	1071	7371
29	1	29	203	1827	23751
Σw	30	4495	100,688	2,136,024	52,451,256
$\Sigma (m)^2$	30	8990	302,064	21,360,240	3,671,587,920

TABLE 2

WEIGHTS FOR EQUALLY SPACED DATA

5 POINTS					6 POINTS				
i	k=1 ρw_1	k=2 ρw_1	k=3 ρw_1	k=4 ρw_1	k=1 ρw_1	k=2 ρw_1	k=3 ρw_1	k=4 ρw_1	i
1	2	2	1	1	5	5	5	1	1
2	3	3	1		8	9	8	1	2
3	3	2			9	9	5		3
4	2				8	5			4
5					5				5
									6
Σw	10	7	2	1	35	28	18	2	

7 POINTS					8 POINTS				
i	k=1 ρw_1	k=2 ρw_1	k=3 ρw_1	k=4 ρw_1	k=1 ρw_1	k=2 ρw_1	k=3 ρw_1	k=4 ρw_1	i
1	3	5	1	3	7	7	7	7	1
2	5	10	2	5	12	15	16	15	2
3	6	12	2	3	15	20	20	15	3
4	6	10	1		16	20	16	7	4
5	5	5			15	15	7		5
6	3				12	7			6
7					7				7
									8
Σw	28	42	6	11	84	84	66	44	

TABLE 2 (Continued)

9 POINTS					10 POINTS				
i	k=1 ρw_1	k=2 ρw_1	k=3 ρw_1	k=4 ρw_1	k=1 ρw_1	k=2 ρw_1	k=3 ρw_1	k=4 ρw_1	i
1	4	28	14	14	9	6	42	18	1
2	7	63	35	35	16	14	112	50	2
3	9	90	50	45	21	21	175	75	3
4	10	100	50	35	24	25	200	75	4
5	10	90	35	14	25	25	175	50	5
6	9	63	14		24	21	112	18	6
7	7	28			21	14	42		7
8	4				16	6			8
9					9				9
									10
Σw	60	462	198	143	165	132	858	286	

11 POINTS					12 POINTS				
i	k=1 ρw_1	k=2 ρw_1	k=3 ρw_1	k=4 ρw_1	k=1 ρw_1	k=2 ρw_1	k=3 ρw_1	k=4 ρw_1	i
1	5	15	30	6	11	55	33	33	1
2	9	36	84	18	20	135	96	105	2
3	12	56	140	30	27	216	168	189	3
4	14	70	175	35	32	280	224	245	4
5	15	75	175	30	35	315	245	245	5
6	15	70	140	18	36	315	224	189	6
7	14	56	84	6	35	280	168	105	7
8	12	36	30		32	216	96	33	8
9	9	15			27	135	33		9
10	5				20	55			10
11					11				11
									12
Σw	110	429	858	143	286	2002	1287	1144	

TABLE 2 (Continued)

13 POINTS					14 POINTS				
i	k=1 ρw_1	k=2 ρw_1	k=3 ρw_1	k=4 ρw_1	k=1 ρw_1	k=2 ρw_1	k=3 ρw_1	k=4 ρw_1	i
1	6	22	11	99	13	13	143	143	1
2	11	55	33	330	24	33	440	495	2
3	15	90	60	630	33	55	825	990	3
4	18	120	84	882	40	75	1200	1470	4
5	20	140	98	980	45	90	1470	1764	5
6	21	147	98	882	48	98	1568	1764	6
7	21	140	84	630	49	98	1470	1470	7
8	20	120	60	330	48	90	1200	990	8
9	18	90	33	99	45	75	825	495	9
10	15	55	11		40	55	440	143	10
11	11	22			33	33	143		11
12	6				24	13			12
13					13				13
									14
Σw	182	1001	572	4862	455	728	9724	9724	

15 POINTS					16 POINTS				
i	k=1 ρw_1	k=2 ρw_1	k=3 ρw_1	k=4 ρw_1	k=1 ρw_1	k=2 ρw_1	k=3 ρw_1	k=4 ρw_1	i
1	7	91	91	1001	15	35	455	1365	1
2	13	234	286	3575	28	91	1456	5005	2
3	18	396	550	7425	39	156	2860	10725	3
4	22	550	825	11550	48	220	4400	17325	4
5	25	675	1050	14700	55	275	5775	23100	5
6	27	756	1176	15876	60	315	6720	26460	6
7	28	784	1176	14700	63	336	7056	26460	7
8	28	756	1050	11550	64	336	6720	23100	8
9	27	675	825	7425	63	315	5775	17325	9
10	25	550	550	3575	60	275	4400	10725	10
11	22	396	286	1001	55	220	2860	5005	11
12	18	234	91		48	156	1456	1365	12
13	13	91			39	91	455		13
14	7				28	35			14
15					15				15
									16
Σw	280	6188	7956	92,378	680	2856	50,388	167,960	

TABLE 2 (Continued)

17 POINTS					18 POINTS				
i	k=1 ρw_1	k=2 ρw_1	k=3 ρw_1	k=4 ρw_1	k=1 ρw_1	k=2 ρw_1	k=3 ρw_1	k=4 ρw_1	i
1	8	40	28	52	17	68	68	68	1
2	15	105	91	195	32	180	224	260	2
3	21	182	182	429	45	315	455	585	3
4	26	260	286	715	56	455	728	1001	4
5	30	330	385	990	65	585	1001	1430	5
6	33	385	462	1188	72	693	1232	1782	6
7	35	420	504	1260	77	770	1386	1980	7
8	36	432	504	1188	80	810	1440	1980	8
9	36	420	462	990	81	810	1386	1782	9
10	35	385	385	715	80	770	1232	1430	10
11	33	330	286	429	77	693	1001	1001	11
12	30	260	182	195	72	585	728	585	12
13	26	182	91	52	65	455	455	260	13
14	21	105	28		56	315	224	68	14
15	15	40			45	180	68		15
16	8				32	68			16
17					17				17
									18
Σw	408	3876	3876	8398	969	7752	11,628	14,212	

TABLE 2 (Continued)

19 POINTS					20 POINTS				
i	k=1 ρw_1	k=2 ρw_1	k=3 ρw_1	k=4 ρw_1	k=1 ρw_1	k=2 ρw_1	k=3 ρw_1	k=4 ρw_1	i
1	9	51	204	612	19	57	969	1938	1
2	17	136	680	2380	36	153	3264	7650	2
3	24	240	1400	5460	51	272	6800	17850	3
4	30	350	2275	9555	64	400	11200	31850	4
5	35	455	3185	14014	75	525	15925	47775	5
6	39	546	4004	18018	84	637	20384	63063	6
7	42	616	4620	20790	91	728	24024	75075	7
8	44	660	4950	21780	96	792	26400	81675	8
9	45	675	4950	20790	99	825	27225	81675	9
10	45	660	4620	18018	100	825	26400	75075	10
11	44	616	4004	14014	99	792	24024	63063	11
12	42	546	3185	9555	96	728	20384	47775	12
13	39	455	2275	5460	91	637	15925	31850	13
14	35	350	1400	2380	84	525	11200	17850	14
15	30	240	680	612	75	400	6800	7650	15
16	24	136	204		64	272	3264	1938	16
17	17	51			51	153	969		17
18	9				36	57			18
19					19				19
									20
Σw	570	6783	42,636	163,438	1330	8778	245,157	653,752	

TABLE 2 (Continued)

21 POINTS					22 POINTS				
i	k=1 ρw_1	k=2 ρw_1	k=3 ρw_1	k=4 ρw_1	k=1 ρw_1	k=2 ρw_1	k=3 ρw_1	k=4 ρw_1	i
1	10	190	285	969	21	35	133	1197	1
2	19	513	969	3876	40	95	456	4845	2
3	27	918	2040	9180	57	171	969	11628	3
4	34	1360	3400	16660	72	255	1632	21420	4
5	40	1800	4900	25480	85	340	2380	33320	5
6	45	2205	6370	34398	96	420	3136	45864	6
7	49	2548	7644	42042	105	490	3822	57330	7
8	52	2808	8580	47190	112	546	4368	66066	8
9	54	2970	9075	49005	117	585	4719	70785	9
10	55	3025	9075	47190	120	605	4840	70785	10
11	55	2970	8580	42042	121	605	4719	66066	11
12	54	2808	7644	34398	120	585	4368	57330	12
13	52	2548	6370	25480	117	546	3822	45864	13
14	49	2205	4900	16660	112	490	3136	33320	14
15	45	1800	3400	9180	105	420	2380	21420	15
16	40	1360	2040	3876	96	340	1632	11628	16
17	34	918	969	969	85	255	969	4845	17
18	27	513	285		72	171	456	1197	18
19	19	190			57	95	133		19
20	10				40	35			20
21					21				21
									22
Σw	770	33,649	86,526	408,595	1771	7084	48,070	624,910	

TABLE 2 (Continued)

23 POINTS					24 POINTS				
i	k=1 ρw_1	k=2 ρw_1	k=3 ρw_1	k=4 ρw_1	k=1 ρw_1	k=2 ρw_1	k=3 ρw_1	k=4 ρw_1	i
1	11	77	77	1463	23	253	1771	253	1
2	21	210	266	5985	44	693	6160	1045	2
3	30	380	570	14535	63	1260	13300	2565	3
4	38	570	969	27132	80	1900	22800	4845	4
5	45	765	1428	42840	95	2565	33915	7752	5
6	51	952	1904	59976	108	3213	45696	11016	6
7	56	1120	2352	76440	119	3808	57120	14280	7
8	60	1260	2730	90090	128	4320	67200	17160	8
9	63	1365	3003	99099	135	4725	75075	19305	9
10	65	1430	3146	102245	140	5005	80080	20449	10
11	66	1452	3146	99099	143	5148	81796	20449	11
12	66	1430	3003	90090	144	5148	80080	19305	12
13	65	1365	2730	76440	143	5005	75075	17160	13
14	63	1260	2352	59976	140	4725	67200	14280	14
15	60	1120	1904	42840	135	4320	57120	11016	15
16	56	952	1428	27132	128	3808	45696	7752	16
17	51	765	969	14535	119	3213	33915	4845	17
18	45	570	570	5985	108	2565	22800	2565	18
19	38	380	266	1463	95	1900	13300	1045	19
20	30	210	77		80	1260	6160	253	20
21	21	77			63	693	1771		21
22	11				44	253			22
23					23				23
									24
Σw 1012					2300	65,780	880,030	197,340	

TABLE 2 (Continued)

25 POINTS					26 POINTS				
i	k=1 ρw_1	k=2 ρw_1	k=3 ρw_1	k=4 ρw_1	k=1 ρw_1	k=2 ρw_1	k=3 ρw_1	k=4 ρw_1	i
1	12	92	506	1518	25	100	2300	2530	1
2	23	253	1771	6325	48	276	8096	10626	2
3	33	462	3850	15675	69	506	17710	26565	3
4	42	700	6650	29925	88	770	30800	51205	4
5	50	950	9975	48450	105	1050	46550	83790	5
6	57	1197	13566	69768	120	1830	63840	122094	6
7	63	1428	17136	91800	133	1596	81396	162792	7
8	68	1632	20400	112200	144	1836	97920	201960	8
9	72	1800	23100	128700	153	2040	112200	235620	9
10	75	1925	25025	139425	160	2200	123200	260260	10
11	77	2002	26026	143143	165	2310	130130	273273	11
12	78	2028	26026	139425	168	2366	132496	273273	12
13	78	2002	25025	128700	169	2366	130130	260260	13
14	77	1925	23100	112200	168	2310	123200	235620	14
15	75	1800	20400	91800	165	2200	112200	201960	15
16	72	1632	17136	69768	160	2040	97920	162792	16
17	68	1428	13566	48450	153	1836	81396	122094	17
18	63	1197	9975	29925	144	1596	63840	83790	18
19	57	950	6650	15675	133	1330	46550	51205	19
20	50	700	3850	6325	120	1050	30800	26565	20
21	42	462	1771	1518	105	770	17710	10626	21
22	33	253	506		88	506	8096	2530	22
23	23	92			69	276	2300		23
24	12				48	100			24
25					25				25
									26

$$\sum w \quad 1300 \quad 26,910 \quad 1,48,005^1 \quad 1,430,715^2 \quad 2925 \quad 16,380 \quad 1,560,780^2 \quad 1,430,715^3$$

¹In computing $m(3)$ in Table 1 the weights used where 2 times these weights.

²In computing $m(3)$ in Table 1 the weights used where 1-1/2 times these weights.

³In computing $m(4)$ in Table 1 the weights used where 2 times these weights.

TABLE 2 (Continued)

27 POINTS					28 POINTS				
1	k=1 ρw_1	k=2 ρw_1	k=3 ρw_1	k=4 ρw_1	k=1 ρw_1	k=2 ρw_1	k=3 ρw_1	k=4 ρw_1	1
1	13	325	130	2990	27	117	585	1755	1
2	25	900	460	12650	52	325	2080	7475	2
3	36	1656	1012	31878	75	600	4600	18975	3
4	46	2530	1771	61985	96	920	8096	37191	4
5	55	3465	2695	102410	115	1265	12397	61985	5
6	63	4410	3724	150822	132	1617	17248	92169	6
7	70	5320	4788	203490	147	1960	22344	125685	7
8	76	6156	5814	255816	160	2200	27360	159885	8
9	81	6885	6732	302940	171	2565	31977	191862	9
10	85	7480	7480	340340	180	2805	35904	216790	10
11	88	7920	8008	364364	187	2992	38896	238238	11
12	90	8190	8281	372645	192	3120	40768	248430	12
13	91	8281	8281	364364	195	3185	41405	248430	13
14	91	8190	8008	340340	196	3185	40768	238238	14
15	90	7920	7480	302940	195	3120	38896	218790	15
16	88	7480	6732	255816	192	2992	35904	191862	16
17	85	6885	5814	203490	187	2805	31977	159885	17
18	81	6156	4788	150822	180	2565	27360	125685	18
19	76	5320	3724	102410	171	2200	22344	92169	19
20	70	4410	2695	61985	160	1960	17248	61985	20
21	63	3465	1771	31878	147	1617	12397	37191	21
22	55	2530	1012	12650	132	1265	8096	18975	22
23	46	1656	460	2990	115	920	4600	7475	23
24	36	900	130		96	600	2080	1755	24
25	25	325			75	325	585		25
26	13				52	117			26
27					27				27
									28
$\sum w$ 1638 118,755 101,790 4,032,015					3654	47,502	525,915 ¹	2,804,880	

¹In computing $m(3)$ in Table 1 the weights used were 3 times these weights.

TABLE 2 (Continued)

29 POINTS					30 POINTS				
i	k=1 ρw_1	k=2 ρw_1	k=3 ρw_1	k=4 ρw_1	k=1 ρw_1	k=2 ρw_1	k=3 ρw_1	k=4 ρw_1	i
1	14	126	819	4095	29	203	1827	23751	1
2	27	351	2925	17550	56	567	6552	102375	2
3	39	650	6500	44850	81	1053	14652	253250	3
4	50	1000	11500	88550	104	1625	26000	523250	4
5	60	1380	17710	148764	125	2250	40250	885500	5
6	69	1771	24794	223146	144	2898	56672	1338876	6
7	77	2156	32340	307230	161	3542	74382	1859550	7
8	84	2520	39900	395010	176	4158	92400	2413950	8
9	90	2850	47025	479655	189	4725	109725	2962575	9
10	95	3135	53295	554268	200	5225	125400	3461175	10
11	99	3366	58344	612612	209	5643	138567	3879876	11
12	102	3536	61880	649740	216	5967	148512	4176900	12
13	104	3640	63700	662480	221	6188	154700	4331600	13
14	105	3675	63700	649740	224	6300	156800	4331600	14
15	105	3640	61880	612612	225	6300	154700	4176900	15
16	104	3536	58344	554268	224	6188	148512	3879876	16
17	102	3366	53295	479655	221	5967	138567	3464175	17
18	99	3135	47025	395010	216	5643	125400	2962575	18
19	95	2850	39900	307230	209	5225	109725	2413950	19
20	90	2520	32340	223146	200	4725	92400	1859550	20
21	84	2156	24794	148764	189	4158	74382	1338876	21
22	77	1771	17710	88550	176	3542	56672	885500	22
23	69	1380	11500	44850	161	2898	40250	523250	23
24	60	1000	6500	17550	144	2250	26000	263250	24
25	50	650	2925	4095	125	1625	14652	102375	25
26	39	351	819		104	1053	6552	23751	26
27	27	126			81	567	1827		27
28	14				56	203			28
29					29				29
									30
Σw 2030 56,637 841,464 7,713,420					4495	100,688	2,136,024	52,451,256	

TABLE 3

BINOMIAL COEFFICIENTS $\binom{r}{k}$

$\begin{array}{c} k \\ r+1 \end{array}$	0	1	2	3	4	5
1	1					
2	1	1				
3	1	2	1			
4	1	3	3	1		
5	1	4	6	4	1	
6	1	5	10	10	5	1
7	1	6	15	20	15	6
8	1	7	21	35	35	21
9	1	8	28	56	70	56
10	1	9	36	84	126	126
11	1	10	45	120	210	252
12	1	11	55	165	330	462
13	1	12	66	220	495	792
14	1	13	78	286	715	1287
15	1	14	91	364	1001	2002
16	1	15	105	455	1365	3003
17	1	16	120	560	1820	4368
18	1	17	136	680	2380	6188
19	1	18	153	816	3060	8568
20	1	19	171	969	3876	11628
21	1	20	190	1140	4845	15504
22	1	21	210	1330	5985	20349
23	1	22	231	1540	7315	26334
24	1	23	253	1771	8855	33649
25	1	24	276	2024	10626	42504
26	1	25	300	2300	12650	53130
27	1	26	325	2600	14950	65780
28	1	27	351	2925	17550	80730
29	1	28	378	3275	20475	98280
30	1	29	406	3654	23751	118755
31	1	30	435	4060	27405	142506
32	1	31	465	4495	31465	169911
33	1	32	496	4960	35960	201376
34	1	33	528	5456	40920	237336
35	1	34	561	5984	46376	278256
36	1	35	595	6545	52360	324632
37	1	36	630	7140	58905	376992
38	1	37	666	7770	66045	435897
39	1	38	703	8436	73815	501942
40	1	39	741	9139	82251	575757

APPENDIX B

APPLICATION OF SMOOTHING METHOD TO
SINE CURVE USING UNEQUALLY SPACED DATA

Given

$$y = \sin x ,$$

with the points unequally spaced over the interval $0 \leq x \leq 1$, use of the divided difference method described in the text yields results tabulated below. First, let

$$y = f_n = A_0P_0 + A_1P_1 + A_2P_2 + \dots + A_nP_n ,$$

where, in this case $n + 1 + m = 11$, $n = 4$ and $P_k = x^k$. In this example $u = x$.

The weights, ω_1 , are chosen by the following modification of the weights already calculated for equally spaced data containing 11 points (see Table 2, Appendix A):

$$\omega_1 = (u_k - u_0) w_1$$

$$\omega_2 = (u_{k+1} - u_1) w_2$$

$$\begin{matrix} \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \end{matrix}$$

$$\omega_1 = (u_{k+i+1} - u_{i+1}) w_1 ,$$

$$\begin{matrix} \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \end{matrix}$$

so that we have

i	w				ω			
	k=4	k=3	k=2	k=1	k=4	k=3	k=2	k=1
1	6	30	15	5	2.28	9.6	3.15	0.40
2	18	84	36	9	7.56	25.2	8.64	1.17
3	30	140	56	12	12.00	40.6	9.52	1.32
4	35	175	70	14	12.60	50.75	12.60	0.84
5	30	175	75	15	13.20	52.5	17.25	1.80
6	18	140	70	15	7.74	44.8	12.60	1.65
7	6	84	56	14	2.34	26.88	11.76	0.98
8		30	36	12		9.6	9.0	1.68
9			15	9			2.70	0.99
10				5				0.35

Then proceeding with the fit we obtain the results:

x	y	\tilde{P}_3 (y - A ₄ x ⁴)	\tilde{P}_2 (\tilde{P}_3 - A ₃ x ³)	\tilde{P}_1 (\tilde{P}_2 - A ₂ x ²)	\tilde{P}_0 (\tilde{P}_1 - A ₁ x)
0.00	0.00000	0.00000	0.00000	0.00000	0.00000
0.08	0.07991	0.07991	0.08000	0.07996	0.00002
0.21	0.20846	0.20842	0.21012	0.20986	0.00001
0.32	0.31457	0.31436	0.32038	0.31977	-0.00001
0.38	0.37092	0.37050	0.38058	0.37972	-0.00001
0.50	0.47943	0.47818	0.50115	0.49966	0.00001
0.61	0.57287	0.57011	0.61182	0.60960	0.00003
0.68	0.62879	0.62452	0.68230	0.67954	0.00002
0.82	0.73115	0.72213	0.82344	0.81943	0.00000
0.93	0.80162	0.78670	0.93450	0.92935	0.00000
1.00	0.84147	0.82152	1.00527	0.99931	0.00001

$$A_4 = 0.01995$$

$$A_3 = -0.18375$$

$$A_2 = 0.00596$$

$$A_1 = 0.99930$$

$$A_0 = 0.00001$$

Therefore

$$\sin x \approx 0.00001 + 0.99930 x + 0.00596 x^2 - 0.18375 x^3 + 0.01995 x^4, \quad 0 \leq x \leq 1 \text{ radian}.$$

APPENDIX C

NOTE ON INTERPOLATION WITH TCHEBYCHEV POLYNOMIALS

When Tchebychev polynomials are used in equation (1) for the purpose of truncating to a lower degree the u should be chosen so as to make the balanced oscillations of $T(u)$ occur within the region of interest. Thus, if accurate results are desired for $x_1 < x < x_j$, u should be chosen so that

$$u = -1 \quad \text{for } x = x_1,$$

and

$$u = 1 \quad \text{for } x = x_j.$$

This is particularly important in the process of inverse interpolation. For example, suppose one is given the following table of values. The problem of finding the solution of $f(x) = 0$ will obviously involve the investigation of a neighborhood near $x = 1.4$.

x	$f(x)$
1.0	-4.0000
1.2	-2.8224
1.4	-0.2464
1.6	4.1216
1.8	10.7136

Linear interpolation using the points (1.4, -0.2464) and (1.6, 4.1216) gives $x = 1.411$ as an approximate solution of $f(x) = 0$. Assume that the desired solution lies between $x = 1.40$ and $x = 1.42$, and choose the linear transformation connecting x and u so that $u = -1$ for $x = 1.40$ and $u = 1$ for $x = 1.42$.

The following table can then be used for determining the A 's in equation (1):

x	u	$T_4(u)$	$T_3(u)$	$T_2(u)$
1.0	-41	22,592,641		
1.2	-21	1,552,321	-36,981	881
1.4	-1	1	-1	1
1.6	19	1,039,681	27,379	721
1.8	39	18,495,361	237,159	

Using these values of Tchebychev polynomials, points on successively lower degree curves, approximating $f(x)$, can be obtained

x	$f(x)$	f_3	$f_2 = f_3 - A_3 T_3$	$f_1 = f_2 - A_2 T_2$
1.0	-4.0000			
1.2	-2.8224	-2.8224	-2.73919275	
1.4	-0.2464	-0.2464	-0.24639775	-0.24753125
1.6	4.1216	4.1216	4.05999725	3.24274375
1.8	10.7136	10.7136		

$$\Delta^4 f = A_4 \Delta^4 T_4, \quad 0.0384 = 30,720,000 A_4$$

$$A_4 = 1.25 \times 10^{-9}$$

Because of the magnitude of A_4 the term $A_4 T_4(u)$ is neglected.

$$\Delta^3 f_3 = A_3 \Delta^3 T_3, \quad 0.432 = 192,000 A_3$$

$$A_3 = 0.00000225$$

$$\Delta^2 f_2 = A_2 \Delta^2 T_2, \quad 1.8136 = 1600 A_2$$

$$A_2 = 0.0011335$$

Linear interpolation with f_1 gives $x = 1.414184$ as an approximate solution of $f(x) = 0$. This can be used as a first approximation for inverse quadratic interpolation in f_2 , and the approximation obtained from f_2 can be used for starting inverse cubic interpolation in f_3 .

In this case

$$f(x) = x^4 + 3x^3 - 2x^2 - 6x$$

and

$$f'(x) = 0$$

for

$$x = \sqrt{2} \approx 1.41421$$

so that

$$f_1$$

has given an excellent approximation of the correct solution.

The accuracy of the various approximations for $f(x)$ within the interval $1.40 < x < 1.42$ for direct interpolation can be seen from the A 's:

$$|f - f_3| \leq |A_4| = 1.25 \times 10^{-9}$$

$$|f - f_2| \leq |A_4| + |A_3| = 1.25 \times 10^{-9} + 2.25 \times 10^{-6}$$

$$|f - f_1| \leq |A_4| + |A_3| + |A_2| \approx 0.00113575$$